

ON THE TYPE OF TRIANGLE GROUPS

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ABSTRACT. We prove a conjecture of R. Schwartz about the type of some complex hyperbolic triangle groups.

1. Introduction

An (n_1, n_2, n_3) -complex reflection complex hyperbolic triangle group is a group of isometries of the complex hyperbolic plane $\mathbb{H}_{\mathbb{C}}^2$ generated by complex reflections I_1, I_2, I_3 in complex geodesics C_1, C_2, C_3 such that C_i and C_{i+1} meet at the angle π/n_i , $n_i \geq 2$ (see Section 2 for definitions). For fixed n_1, n_2, n_3 , modulo conjugacy in $\text{Isom}(\mathbb{H}_{\mathbb{C}}^2)$, there exists in general a 1-parameter family of (n_1, n_2, n_3) -triangle groups. Assume $n_1 \leq n_2 \leq n_3$, $n_i \in \mathbb{N}$. The triple (n_1, n_2, n_3) is classified with respect to the behavior of the isometries

$$W_A := I_3 I_2 I_1 I_2 \quad \text{and} \quad W_B := I_1 I_2 I_3$$

while a parameter of the (n_1, n_2, n_3) -triangle group family varies in a canonical way. The triple is said to be of type A if W_A becomes regular elliptic before W_B and of type B if W_B becomes regular elliptic before W_A (see Subsections 2.2 and 2.3).

In this paper, we prove the following

1.1. Conjecture [Sch2, Conjecture 5.2]. *The triple (n_1, n_2, n_3) has type A if $n_1 \leq 9$ and type B if $n_1 \geq 14$.*

Conjecture 1.1 is a tiny portion of a complete conjectural picture [Sch2] that describes (n_1, n_2, n_3) -triangle groups with focus on discreteness. Roughly speaking, the picture is as follows: An (n_1, n_2, n_3) -triangle group is discrete if neither W_A nor W_B is regular elliptic. When (n_1, n_2, n_3) has type B , then the converse also holds. If (n_1, n_2, n_3) has type A , then there is a countable collection of ‘extra’ discrete groups. While sorting triangle groups into types A and B is (as we shall see) a matter of a couple of simple tricks intended to avoid the huge amount of calculus a straightforward approach leads to, classifying those groups with respect to discreteness is certainly a much more difficult and interesting task. It has been accomplished for (∞, ∞, ∞) -triangle groups in [GoP] and [Sch1] (see also [Sch3]) and for (n_1, n_2, n_3) -triangle groups with sufficiently large n_1 in [Sch4].

In principle, we could say that our proof is computer independent. In fact, we use the computer only to obtain approximate values of the cosine function needed to establish some inequalities that hold by a wide margin of error (Lemmas 3.1 and 3.2). The inequalities proved in this way are marked with the symbols \prec and \succ .

Conjecture 1.1 has been solved for sufficiently large n_1 in [Sch4] and for triples of the form (n, n, ∞) in [W-G]. In [Pra], it is shown that the triples corresponding to triangles with $r_1^2 + r_2^2 + r_3^2 - 1 = 2r_1 r_2 r_3$ and $r_1 r_2 r_3 \geq \frac{13+\sqrt{297}}{32}$ (see [Pra] and also Subsection 2.2) are of type B .

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2. Preliminaries

2.1. Basic background. Let V be a 3-dimensional \mathbb{C} -vector space equipped with a hermitian form $\langle -, - \rangle$ of signature $+, +, -$. The complex hyperbolic plane $\mathbb{H}_{\mathbb{C}}^2$ can be identified with the open 4-ball¹

$$BV := \{p \in \mathbb{P}_{\mathbb{C}}V \mid \langle p, p \rangle < 0\}.$$

The complex hyperbolic distance $d(p_1, p_2)$ between two points $p_1, p_2 \in BV$ is given in terms of the *tance*²

$$\text{ta}(p_1, p_2) := \frac{\langle p_1, p_2 \rangle \langle p_2, p_1 \rangle}{\langle p_1, p_1 \rangle \langle p_2, p_2 \rangle}$$

by $\cosh^2(d(p_1, p_2)/2) = \text{ta}(p_1, p_2)$ [Gol, p. 77]. The ideal boundary of $\mathbb{H}_{\mathbb{C}}^2$ is the 3-sphere

$$SV := \{p \in \mathbb{P}_{\mathbb{C}}V \mid \langle p, p \rangle = 0\}$$

formed by the *isotropic* points in $\mathbb{P}_{\mathbb{C}}V$. Notice that the tance $\text{ta}(p_1, p_2)$ is well-defined for all nonisotropic $p_1, p_2 \in \mathbb{P}_{\mathbb{C}}V$. We put $\overline{BV} := BV \cup SV$.

Every projective line L in $\mathbb{P}_{\mathbb{C}}V$ has the form $L = \mathbb{P}_{\mathbb{C}}p^{\perp}$, where $p \in \mathbb{P}_{\mathbb{C}}V$ and $p^{\perp} = \{v \in V \mid \langle v, p \rangle = 0\}$. We call p the *polar point* to L . If $p \notin \overline{BV}$, then $\mathbb{P}_{\mathbb{C}}p^{\perp} \cap \overline{BV}$ is a *complex geodesic*. Two distinct complex geodesics C_1, C_2 are *concurrent* (respectively, *asymptotic*, *ultraparallel*) if and only if $C_1 \cap C_2 \in BV$ (respectively, $C_1 \cap C_2 \in SV$, $C_1 \cap C_2 = \emptyset$).

2.1.1. Lemma [Gol, p. 100]. *Two distinct complex geodesics C_1, C_2 with polar points p_1, p_2 are concurrent, asymptotic, ultraparallel if and only if $\text{ta}(p_1, p_2) < 1$, $\text{ta}(p_1, p_2) = 1$, $\text{ta}(p_1, p_2) > 1$, respectively. If $\text{ta}(p_1, p_2) \leq 1$, then the angle $\angle(C_1, C_2) \in [0, \pi/2]$ between C_1 and C_2 is given by $\cos^2 \angle(C_1, C_2) = \text{ta}(p_1, p_2)$ ■*

Given $p \notin \overline{BV}$, define $I \in \text{SU } V$ by the rule

$$(2.1.2) \quad I : x \mapsto 2 \frac{\langle x, p \rangle}{\langle p, p \rangle} p - x.$$

The corresponding isometry in $\text{PU } V$ is known as the *complex reflection* in the complex geodesic $\mathbb{P}_{\mathbb{C}}p^{\perp} \cap \overline{BV}$. For brevity, we will call it simply the *reflection* in the complex geodesic in question.

2.2. Complex hyperbolic triangles. A *complex hyperbolic triangle* is a triple (C_1, C_2, C_3) of complex geodesics in \overline{BV} . Each complex geodesic C_i is a *side* of the triangle. If the sides C_i and C_{i+1} meet at the angle π/n_i , where $n_i \geq 2$ (we allow n_i to be infinite, meaning that C_i and C_{i+1} are asymptotic or equal), the triangle (C_1, C_2, C_3) is referred to as an (n_1, n_2, n_3) -*triangle*. We call an (n_1, n_2, n_3) -triangle (C_1, C_2, C_3) *nondegenerate* if the form restricted to the subspace $\mathbb{C}p_1 + \mathbb{C}p_2 + \mathbb{C}p_3$ of V is nondegenerate, being p_i the polar point to C_i .

2.2.1. Lemma (compare with [Pra, Proposition 1]). *Let (C_1, C_2, C_3) be a nondegenerate (n_1, n_2, n_3) -triangle with $n_i > 2$, $i = 1, 2, 3$. Denote by p_i the polar point to C_i . Define*

$$r_i := \sqrt{\text{ta}(p_i, p_{i+1})} = \cos \frac{\pi}{n_i} > 0, \quad \varkappa := \frac{\langle p_1, p_2 \rangle \langle p_2, p_3 \rangle \langle p_3, p_1 \rangle}{\langle p_1, p_1 \rangle \langle p_2, p_2 \rangle \langle p_3, p_3 \rangle}, \quad \varepsilon := \frac{\varkappa}{|\varkappa|}, \quad t := \text{Re } \varepsilon.$$

¹The symbol $:=$ stands for ‘equals by definition.’

²Here, and in what follows, we frequently do not distinguish the notation of a point in $\mathbb{P}_{\mathbb{C}}V$ and of a chosen representative of it in V when a concept or expression does not depend on such a choice.

Then, the numbers r_i and t constitute a complete set of geometrical invariants of (C_1, C_2, C_3) . They satisfy $0 < r_i \leq 1$, $|t| \leq 1$, and

$$(2.2.2) \quad 1 + 2r_1r_2r_3t - (r_1^2 + r_2^2 + r_3^2) \leq 0.$$

All values of r_i and t subject to the conditions $0 < r_i \leq 1$, $|t| \leq 1$, and $1 + 2r_1r_2r_3t - (r_1^2 + r_2^2 + r_3^2) \leq 0$ correspond to a nondegenerate (n_1, n_2, n_3) -triangle with $n_i > 2$.

Proof. The numbers r_i and ε are invariant under the action of $\text{PU } V$ on the triple (p_1, p_2, p_3) . Choosing suitable representatives $p_i \in V$, we can assume that

$$(2.2.3) \quad \begin{pmatrix} 1 & r_1 & r_3\bar{\varepsilon} \\ r_1 & 1 & r_2 \\ r_3\varepsilon & r_2 & 1 \end{pmatrix}$$

is the Gram matrix of (p_1, p_2, p_3) . If the triples (p_1, p_2, p_3) and (p'_1, p'_2, p'_3) have the same Gram matrix and if the hermitian form is nondegenerate being restricted to the subspaces generated by p_1, p_2, p_3 and by p'_1, p'_2, p'_3 , then there exists $I \in \text{U } V$ such that $Ip_i = p'_i$. The triangles corresponding to $(r_1, r_2, r_3, \varepsilon)$ and to $(r'_1, r'_2, r'_3, \varepsilon')$ differ by an anti-holomorphic isometry of $\mathbb{H}_{\mathbb{C}}^2$ if and only if $r_i = r'_i$ and $\varepsilon = \bar{\varepsilon}'$. The rest follows from Sylvester's criterion ■

From now on, all (n_1, n_2, n_3) -triangles are assumed to be nondegenerate.

2.2.4. Remark. For fixed $3 < n_1 \leq n_2 \leq n_3$, there exists a non-empty 1-parameter family of (n_1, n_2, n_3) -triangles. Indeed, the left-hand side of the inequality (2.2.2) is increasing in t and, hence, attains its minimum value at $t = -1$. We have

$$1 - 2r_1r_2r_3 - (r_1^2 + r_2^2 + r_3^2) < 1 - 2\cos^3 \frac{\pi}{3} - 3\cos^2 \frac{\pi}{3} = 0 \blacksquare$$

Let $3 < n_1 \leq n_2 \leq n_3$. In the terms of Lemma 2.2.1, define

$$(2.2.5) \quad t_M := \frac{r_1^2 + r_2^2 + r_3^2 - 1}{2r_1r_2r_3}, \quad t_{max} := \min\{t_M, 1\}.$$

The *canonical* path of deformation of the (n_1, n_2, n_3) -triangle family is the one that starts with $t = -1$ and ends with $t = t_{max}$.

2.3. Complex hyperbolic triangle groups. The subgroup in $\text{PU } V$ generated by the reflections in the sides of an (n_1, n_2, n_3) -triangle is called an (n_1, n_2, n_3) -*triangle group*. Up to conjugacy in $\text{Isom}(\mathbb{H}_{\mathbb{C}}^2)$, all nondegenerate (n_1, n_2, n_3) -triangle groups are described by Lemma 2.2.1.

Fix $3 < n_1 \leq n_2 \leq n_3$ and assume $n_i \in \mathbb{N}$. For a given (n_1, n_2, n_3) -triangle (C_1, C_2, C_3) , define

$$W_A := I_3I_2I_1I_2, \quad W_B := I_1I_2I_3,$$

where I_i stands for the reflection in C_i . The triple (n_1, n_2, n_3) is characterized with respect to the behavior of the isometries W_A and W_B during the canonical deformation of the 1-parameter family of (n_1, n_2, n_3) -triangle groups. Specifically, (n_1, n_2, n_3) is said to be of *type A* if W_A becomes regular elliptic before W_B and of *type B* if W_B becomes regular elliptic before W_A (see [Gol] for the classification of holomorphic isometries of $\mathbb{H}_{\mathbb{C}}^2$).

2.3.1. Lemma (compare with [Pra, Proposition 12]). *W_A is always hyperbolic at the beginning of the deformation. In the terms of Lemma 2.2.1, W_A is regular elliptic if and only if*

$$t > t_{W_A} := \frac{r_3^2 + 4r_1^2r_2^2 - 1}{4r_1r_2r_3}.$$

Proof. Let (C_1, C_2, C_3) be an (n_1, n_2, n_3) -triangle and let p_i denote the polar point to C_i . The isometry W_A is the product of two reflections: one in the complex geodesic with polar point $I_2 p_1$ and the other in C_3 . The nature of W_A is hence determined by the relative position of these complex geodesics. Taking (2.2.3) as the Gram matrix of suitable representatives $p_i \in V$ and applying (2.1.2), we obtain

$$\text{ta}(I_2 p_1, p_3) = \frac{\langle I_2 p_1, p_3 \rangle \langle p_3, I_2 p_1 \rangle}{\langle I_2 p_1, I_2 p_1 \rangle \langle p_3, p_3 \rangle} = \left| \langle 2r_1 p_2 - p_1, p_3 \rangle \right|^2 = 4r_1^2 r_2^2 - 4r_1 r_2 r_3 t + r_3^2.$$

At the beginning of the deformation,

$$\text{ta}(I_2 p_1, p_3) = 4r_1^2 r_2^2 + 4r_1 r_2 r_3 + r_3^2 > 4(\cos^4 \frac{\pi}{3} + \cos^3 \frac{\pi}{3}) + \cos^2 \frac{\pi}{3} = 1.$$

By Lemma 2.1.1, this implies that W_A is hyperbolic. It remains to observe that $\text{ta}(I_2 p_1, p_3)$ is decreasing in t and that, by Lemma 2.1.1, W_A becomes parabolic exactly when $t = t_{W_A}$ ■

In order to deal with W_B , we need the following

2.3.2. Lemma [Gol, Theorem 6.2.4]. *Define a map $f : \mathbb{C} \rightarrow \mathbb{R}$ by*

$$f(z) := |z|^4 - 8 \operatorname{Re}(z^3) + 18|z|^2 - 27.$$

Given $J \in \operatorname{PU} V$, let $\hat{J} \in \operatorname{SU} V$ be a lift of J . Then, J is regular elliptic (respectively, loxodromic) if and only if $f(\operatorname{tr} \hat{J}) < 0$ (respectively, $f(\operatorname{tr} \hat{J}) > 0$). The pre-image $f^{-1}(0) \subset \mathbb{C}$ is a deltoid. If $\operatorname{tr} \hat{J} \in \mathbb{R}$, then J is loxodromic if and only if $\operatorname{tr} \hat{J} \notin [-1, 3]$ ■

Take the lift $W_B \in \operatorname{SU} V$ determined by the lifts of I_1, I_2, I_3 in (2.1.2). The trace $\tau := \operatorname{tr} W_B$ is given by

$$(2.3.3) \quad \tau := \operatorname{tr} W_B = 8r_1 r_2 r_3 \varepsilon - 4(r_1^2 + r_2^2 + r_3^2) + 3 \in \mathbb{C}$$

(see, for instance, [Pra]).

2.3.4. Lemma. W_B is always loxodromic at the beginning of the deformation.

Proof. By (2.3.3),

$$\tau = -8r_1 r_2 r_3 - 4(r_1^2 + r_2^2 + r_3^2) + 3 < -8 \cos^3 \frac{\pi}{3} - 12 \cos^2 \frac{\pi}{3} + 3 = -1$$

at the beginning of the deformation. The result follows from Lemma 2.3.2 ■

During the deformation, τ belongs to the circle

$$(2.3.5) \quad F := \left\{ (x, y) \in \mathbb{C} \mid (x + 4(r_1^2 + r_2^2 + r_3^2) - 3)^2 + y^2 = (8r_1 r_2 r_3)^2 \right\},$$

where $x := \operatorname{Re} \tau$ and $y := \operatorname{Im} \tau$. By Lemma 2.2.1, we can assume that $\operatorname{Im} \varepsilon \geq 0$, i.e., $y \geq 0$. By (2.3.3), the coordinate $x = \operatorname{Re} \tau$ and the parameter $t = \operatorname{Re} \varepsilon$ are linked by the formula

$$(2.3.6) \quad t = \frac{x + 4(r_1^2 + r_2^2 + r_3^2) - 3}{8r_1 r_2 r_3}.$$

Hence, we can think of the parameter t as ‘living’ in the upper half-circle of F .

3. Proof of the conjecture

In what follows, we will refer to the elementary Lemmas 4.1 and 4.3, proved in Section 4.

3.1. Proposition. *If $14 \leq n_1 \leq n_2 \leq n_3$, then the triple (n_1, n_2, n_3) is of type B.*

Proof. In the terms of Lemma 2.2.1, $\cos \frac{\pi}{14} \leq r_1 \leq r_2 \leq r_3 \leq 1$.

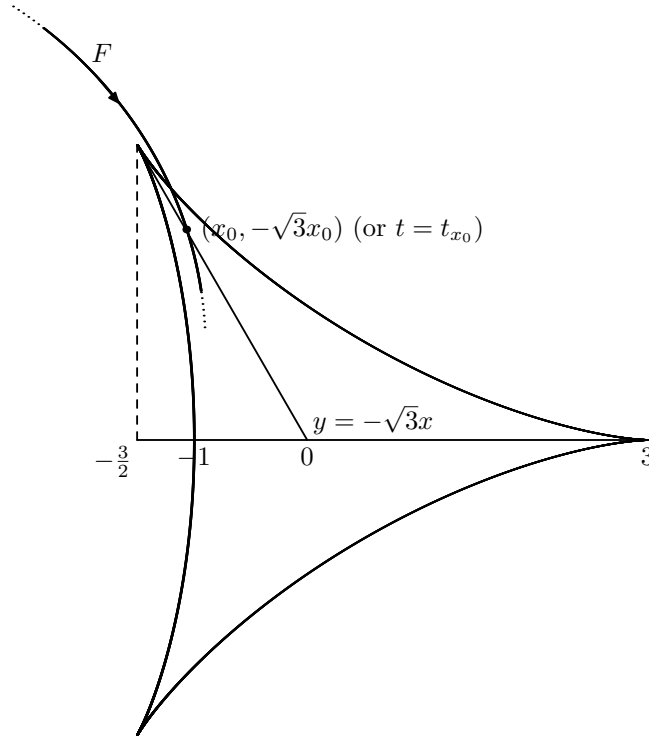
Intersection points $(x, -\sqrt{3}x) \in \mathbb{C}$ of the line $y = -\sqrt{3}x$ that passes through the vertex $(-\frac{3}{2}, \frac{3\sqrt{3}}{2})$ and through the center $(0, 0)$ of Goldman's deltoid (Lemma 2.3.2) with the circle F given by (2.3.5) satisfy³

$$4x^2 + 2(4 \sum r_i^2 - 3)x + (4 \sum r_i^2 - 3)^2 - (8r_1r_2r_3)^2 = 0.$$

By Lemma 4.1 (1), the discriminant

$$D_1 := 4(-3(4 \sum r_i^2 - 3)^2 + (16r_1r_2r_3)^2)$$

of the above equation is such that $D_1 > 0$. Take the root $x_0 := \frac{-2(4 \sum r_i^2 - 3) + \sqrt{D_1}}{8}$.



We will show that $x_0 \in (-\frac{3}{2}, -1)$. In particular, this implies that $(x_0, -\sqrt{3}x_0) \in F$ is ‘inside of’ Goldman's deltoid.⁴ The inequality $x_0 < -1$, equivalent to $\sqrt{D_1} < 2(4 \sum r_i^2 - 3) - 8$, follows from Lemma 4.1 (2) and from⁵

$$2(4 \sum r_i^2 - 3) - 8 \geq 2(12 \cos^2 \frac{\pi}{14} - 3) - 8 > 8.8.$$

³Obviously, we always sum over $i = 1, 2, 3$.

⁴It would suffice to prove here a weaker inequality, but we will need later the fact that $x_0 < -1$.

⁵As stated in the introduction, the symbols $<$ and $>$ are used for the inequalities proved using the computer to find approximate values of the cosine function.

The inequality $x_0 > -\frac{3}{2}$ follows from $-(4 \sum r_i^2 - 3)^2 + 3(4 \sum r_i^2 - 3) + (8r_1r_2r_3)^2 - 9 > 0$ which is a consequence of Lemma 4.1 (3).

According to (2.3.6), the value of the deformation parameter t that corresponds to x_0 is

$$t_{x_0} := \frac{x_0 + 4 \sum_i r_i^2 - 3}{8r_1r_2r_3}.$$

It satisfies $t_{x_0} \in (-1, t_{max})$, being t_{max} as defined in (2.2.5). Indeed, the inequality $t_{x_0} > -1$ is straightforward. The inequality $t_{x_0} < 1$, equivalent to $\sqrt{D_1} < -6(4 \sum r_i^2 - 3) + 64r_1r_2r_3$, follows from Lemma 4.1 (2), (4). Finally, the inequality $t_{x_0} < t_M$ is equivalent to $x_0 < -1$.

We have just proved that the deformation parameter assumes the value $t = t_{x_0}$. By Lemma 2.3.2, the isometry W_B is regular elliptic when $t = t_{x_0}$ since $(x_0, -\sqrt{3}x_0)$ is inside of the deltoid. By Lemma 2.3.1, in order to show that W_B becomes regular elliptic before W_A , it suffices to show that $t_{x_0} < t_{W_A}$. This follows from

$$3(4 \sum r_i^2 - 3)^2 - 12(r_3^2 + 4r_1^2r_2^2 - 1)(4 \sum r_i^2 - 3) + 16(r_3^2 + 4r_1^2r_2^2 - 1)^2 - (8r_1r_2r_3)^2 > 0$$

which is a consequence of Lemma 4.1 (5) ■

3.5. Proposition. *If $n_1 \leq n_2 \leq n_3$ and $4 \leq n_1 \leq 9$, then the triple (n_1, n_2, n_3) is of type A.*

Proof. In the terms of Lemma 2.2.1, $r_1 \leq r_2 \leq r_3 \leq 1$ and $\cos \frac{\pi}{4} \leq r_1 \leq \cos \frac{\pi}{9}$.

First, let us show that the deformation parameter t assumes the value $t = t_{W_A}$, being t_{W_A} as in Lemma 2.3.1. In other words, we need to show that $t_{W_A} \in [-1, t_{max}]$ (see 2.2.5). The inequalities $t_{W_A} > -1$ and $t_{W_A} \leq 1$ are straightforward. The inequality $t_{W_A} \leq t_M$, equivalent to $2(r_1^2 + r_2^2 - 2r_1^2r_2^2) + r_3^2 - 1 \geq 0$, is a consequence of Lemma 4.3 (1).

Assume $\cos \frac{\pi}{4} \leq r_1 \leq \cos \frac{\pi}{8}$.

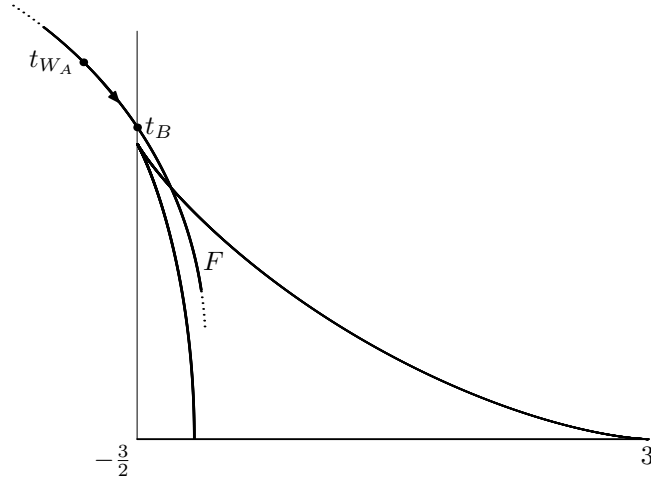
At the beginning $t = -1$ of the deformation, the trace τ of W_B given by (2.3.3) satisfies

$$\tau \leq -8 \cos^3 \frac{\pi}{4} - 12 \cos^2 \frac{\pi}{4} + 3 < -\frac{3}{2}.$$

By Lemma 2.3.2, this means that W_B may become elliptic only after the parameter

$$t_B := \frac{-\frac{3}{2} + 4 \sum r_i^2 - 3}{8r_1r_2r_3}$$

that corresponds, by (2.3.6), to $x = -\frac{3}{2}$.



The inequality $t_{W_A} < t_B$ is equivalent to $8(r_1^2 + r_2^2 - 2r_1^2 r_2^2) + 4r_3^2 - 5 > 0$ and follows from Lemma 4.3 (2). This implies that (n_1, n_2, n_3) is of type A.

We now consider the case $r_1 = \cos \frac{\pi}{9}$.

By Lemma 2.3.2, intersection points of the deltoid with the line $l := \{(x, y) \in \mathbb{C} \mid y = \frac{3\sqrt{3}}{5}(1 - x)\}$ satisfy $(2x + 3)^2(169x^2 - 158x - 111) = 0$. The roots of this equation are $x = -\frac{3}{2}$ (that corresponds to a vertex of the deltoid),

$$x = x_1 := \frac{1}{169}(79 - 50\sqrt{10}), \quad \text{and} \quad x = \frac{1}{169}(79 + 50\sqrt{10}).$$

Intersection points of the circle F given by (2.3.5) with the line l satisfy

$$52x^2 + 2(25(4 \sum r_i^2 - 3) - 27)x + 25((4 \sum r_i^2 - 3)^2 - (8r_1 r_2 r_3)^2) + 27 = 0.$$

By Lemma 4.3 (3), the discriminant

$$D_2 := 100(-27(4 \sum r_i^2 - 3)^2 - 54(4 \sum r_i^2 - 3) + 52(8r_1 r_2 r_3)^2 - 27)$$

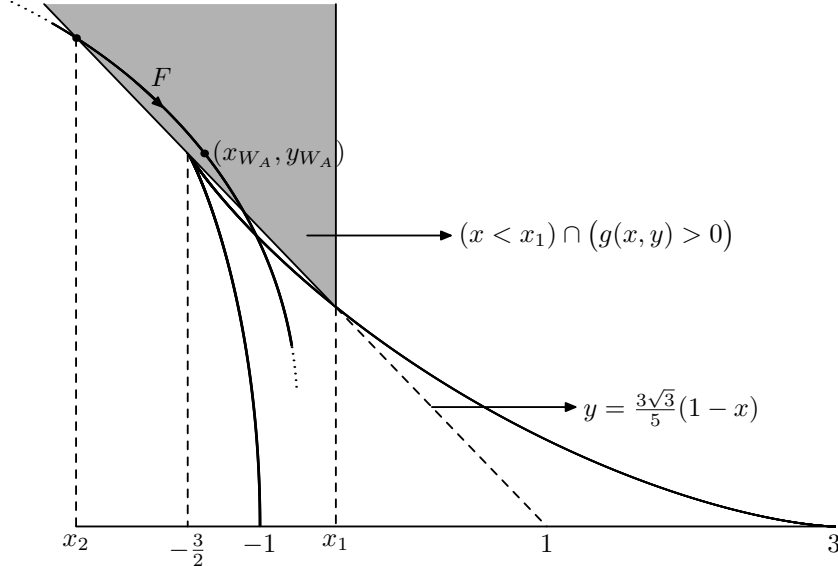
of the above equation is such that $D_2 > 0$. Take the root

$$x_2 := \frac{-2(25(4 \sum r_i^2 - 3) - 27) - \sqrt{D_2}}{104}.$$

In order to prove that $(9, n_2, n_3)$ is of type A, it suffices to apply Lemma 2.3.1 after showing the following facts (see the picture below):

(1) $x_2 < -\frac{3}{2}$. This implies that, when F crosses l for the first time (thus entering the region in grey), W_B has not become elliptic yet.

(2) $x_{W_A} < x_1$ and $g(x_{W_A}, y_{W_A}) > 0$, where $(x_{W_A}, y_{W_A}) \in F$ is the point that corresponds to t_{W_A} by (2.3.6) and $g(x, y) := y - \frac{3\sqrt{3}}{5}(1 - x)$. This implies that we are still in the grey region when $t = t_{W_A}$. In particular, W_B has not become elliptic yet.



The inequality $x_2 < -\frac{3}{2}$, equivalent to $\sqrt{D_2} > -50(4\sum r_i^2 - 3) + 210$, follows from

$$-50(4\sum r_i^2 - 3) + 210 \leq -50(12\cos^2 \frac{\pi}{9} - 3) + 210 < 0.$$

By (2.3.6) and (2.3.5),

$$x_{W_A} := 2(r_3^2 + 4r_1^2 r_2^2 - 1) - (4\sum r_i^2 - 3), \quad y_{W_A} := 2\sqrt{(4r_1 r_2 r_3)^2 - (r_3^2 + 4r_1^2 r_2^2 - 1)^2}.$$

The inequality $x_{W_A} < x_1$, equivalent to $2(r_3^2 + 4r_1^2 r_2^2 - 1) - (4\sum r_i^2 - 3) - \frac{1}{169}(79 - 50\sqrt{10}) < 0$, follows from Lemma 4.3 (4). Finally, $g(x_{W_A}, y_{W_A}) > 0$ is a consequence of

$$25((4r_1 r_2 r_3)^2 - (r_3^2 + 4r_1^2 r_2^2 - 1)^2) - 27(2r_2^2(1 - 2r_1^2) + 2r_1^2 + r_3^2)^2 > 0$$

which follows from Lemma 4.3 (5) ■

3.6. Remark. For the sake of generality, we have not considered yet the $(3, n_2, n_3)$ -triangles, where $3 \leq n_2 \leq n_3 \leq \infty$. As in Remark 2.2.4, it is easy to see that there exists in general a non-empty one parameter family of $(3, n_2, n_3)$ -triangles. The only exception is the $(3, 3, 3)$ -triangle, which is rigid. Proceeding as in the first part of the proof of Proposition 3.5 (where we dealt with the $4 \leq n_1 \leq 8$ cases) and using the fact that $n_i \in \mathbb{N}$, one easily shows that the non-rigid $(3, n_1, n_2)$ -triangles are of type A ■

4. Taking Derivatives

4.1. Lemma. Suppose that $\cos \frac{\pi}{14} \leq x, y, z \leq 1$. Define

$$\begin{aligned} f_1(x, y, z) &:= -3(4(x^2 + y^2 + z^2) - 3)^2 + (16xyz)^2, \\ f_2(x, y, z) &:= -(4(x^2 + y^2 + z^2) - 3)^2 + 3(4(x^2 + y^2 + z^2) - 3) + (8xyz)^2 - 9, \\ f_3(x, y, z) &:= -3(4(x^2 + y^2 + z^2) - 3) + 32xyz, \\ f_4(x, y, z) &:= 3(4(x^2 + y^2 + z^2) - 3)^2 - 12(z^2 + 4x^2 y^2 - 1)(4(x^2 + y^2 + z^2) - 3) + 16(z^2 + 4x^2 y^2 - 1)^2 - (8xyz)^2. \end{aligned}$$

Then,

$$\begin{aligned} \text{(1)} \quad 31 < 4f_1(x, y, z) \leq 52, \quad \text{(2)} \quad 5.5 < \sqrt{4f_1(x, y, z)} < 7.3, \quad \text{(3)} \quad f_2(x, y, z) > 0.5, \\ \text{(4)} \quad 2f_3(x, y, z) > 8.8, \quad \text{and} \quad \text{(5)} \quad f_4(x, y, z) > 0.1. \end{aligned}$$

Proof is straightforward. We show, for instance, the first and the last items. Notice that

$$(4.2) \quad 8\cos^2 \frac{\pi}{14} - 3 > 0.$$

(1) Taking derivatives,

$$\frac{\partial f_1}{\partial x} = 16x(-3(4(x^2 + y^2 + z^2) - 3) + 32y^2 z^2), \quad \frac{\partial^2 f_1}{\partial x^2} = 16(-36x^2 - 3(4(y^2 + z^2) - 3) + 32y^2 z^2).$$

By (4.2), $\frac{\partial^2 f_1}{\partial x^2}$ is increasing in y and in z . Hence, $\frac{\partial^2 f_1}{\partial x^2} \leq \frac{\partial^2 f_1}{\partial x^2}(\cos \frac{\pi}{14}, 1, 1) < 0$. This implies that $\frac{\partial f_1}{\partial x}$ is decreasing in x and (4.2) implies that it is increasing in y and in z . So, $\frac{\partial f_1}{\partial x} \geq \frac{\partial f_1}{\partial x}(1, \cos \frac{\pi}{14}, \cos \frac{\pi}{14}) > 0$. We have just proved that f_1 is increasing in every variable. It follows that

$$31 < 4f_1(\cos \frac{\pi}{14}, \cos \frac{\pi}{14}, \cos \frac{\pi}{14}) \leq 4f_1 \leq 4f_1(1, 1, 1) \leq 52.$$

(5) Taking derivatives,

$$\frac{\partial f_4}{\partial x} = 16x(4x^2(16y^4 - 12y^2 + 3) - 2y^2(12y^2 + 8z^2 - 7) + 6z^2 - 3),$$

$$\frac{\partial^2 f_4}{\partial x \partial y} = 64xy(-24(x^2 + y^2) + 64x^2y^2 - 8z^2 + 7), \quad \frac{\partial f_4}{\partial z} = 8z(12(x^2 + y^2) - 32x^2y^2 + 8z^2 - 5).$$

Put $g(y) := 16y^4 - 12y^2 + 3$. Notice that $g'(y) > 0 \iff 8y^2 - 3 > 0$ and that the last inequality follows from (4.2). Hence, $g(y) \geq g(\cos \frac{\pi}{14}) \succ 0$. This implies that $\frac{\partial f_4}{\partial x}$ is increasing in x . By (4.2), $\frac{\partial^2 f_4}{\partial x \partial y}$ is increasing in x and in y . Hence, $\frac{\partial^2 f_4}{\partial x \partial y} \geq \frac{\partial^2 f_4}{\partial x \partial y}(\cos \frac{\pi}{14}, \cos \frac{\pi}{14}, 1) \succ 0$. In other words, $\frac{\partial f_4}{\partial x}$ is increasing also in y . It is decreasing in z by (4.2). So, $\frac{\partial f_4}{\partial x} \geq \frac{\partial f_4}{\partial x}(\cos \frac{\pi}{14}, \cos \frac{\pi}{14}, 1) \succ 0$. It follows that f_4 is increasing in both x and y . Moreover, $\frac{\partial f_4}{\partial z}$ is increasing in z and decreasing in x and in y by (4.2). This implies that $\frac{\partial f_4}{\partial z} \leq \frac{\partial f_4}{\partial z}(\cos \frac{\pi}{14}, \cos \frac{\pi}{14}, 1) \prec 0$, that is, f_4 is decreasing in z . Finally,

$$f_4 \geq f_4(\cos \frac{\pi}{14}, \cos \frac{\pi}{14}, 1) \succ 0.1 \blacksquare$$

4.3. Lemma. Define

$$g_1(x, y, z) := 2(x^2 + y^2 - 2x^2y^2) + z^2 - 1 \text{ for } \cos \frac{\pi}{4} \leq x \leq y \leq z \leq 1,$$

$$g_2(x, y, z) := 8(x^2 + y^2 - 2x^2y^2) + 4z^2 - 5 \text{ for } \cos \frac{\pi}{4} \leq x \leq \cos \frac{\pi}{8} \text{ and } x \leq y \leq z \leq 1.$$

Suppose that $x = \cos \frac{\pi}{9}$ and that $\cos \frac{\pi}{9} \leq y \leq z \leq 1$. Define

$$g_3(y, z) := -27(4(x^2 + y^2 + z^2) - 3)^2 - 54(4(x^2 + y^2 + z^2) - 3) + 52(8xyz)^2 - 27,$$

$$g_4(y, z) := 2(z^2 + 4x^2y^2 - 1) - (4(x^2 + y^2 + z^2) - 3) - \frac{1}{169}(79 - 50\sqrt{10}),$$

$$g_5(y, z) := 25((4xyz)^2 - (z^2 + 4x^2y^2 - 1)^2) - 27(2y^2(1 - 2x^2) + 2x^2 + z^2)^2.$$

Then,

$$(1) \quad g_1(x, y, z) \geq 0, \quad (2) \quad g_2(x, y, z) \succ 0.1, \quad (3) \quad g_3(y, z) \succ 296$$

$$(4) \quad g_4(y, z) \prec -0.9, \quad \text{and} \quad (5) \quad g_5(y, z) \succ 0.2.$$

Proof is straightforward. We show the first and the last items, for instance.

(1) Clearly, $g_1(x, y, z) \geq g_1(x, y, y)$. Since $g_1(x, y, y) = 2x^2(1 - 2y^2) + 3y^2 - 1$ and $1 - 2y^2 \leq 0$, we obtain $g_1(x, y, z) \geq g_1(x, y, y) \geq g_1(y, y, y) = -4y^4 + 5y^2 - 1 \geq 0$.

(5) Taking derivatives,

$$\frac{\partial g_5}{\partial y} = 8y(z^2(104x^2 - 27) + 2y^2(-100x^4 - 27(1 - 2x^2)^2) - 4x^2(1 - 27x^2)).$$

It follows from $104x^2 - 27 \succ 0$ and $-100x^4 - 27(1 - 2x^2)^2 < 0$ that $\frac{\partial g_5}{\partial y} \leq \frac{\partial g_5}{\partial y}(\cos \frac{\pi}{9}, 1) \prec 0$. In other words, $g_5(y, z)$ is decreasing in y . Hence, $g_5(y, z) \geq g_5(z, z)$. Define

$$h(z) := g_5(z, z) = 25(16x^2z^4 - (z^2 + 4x^2z^2 - 1)^2) - 27(z^2(3 - 4x^2) + 2x^2)^2.$$

We have

$$\frac{\partial h}{\partial z} = 4z(4z^2(-208x^4 + 212x^2 - 67) + 25(1 + 4x^2) - 54x^2(3 - 4x^2)).$$

It follows from $-208x^4 + 212x^2 - 67 < 0$ that $\frac{\partial h}{\partial z} \leq \frac{\partial h}{\partial z}(\cos \frac{\pi}{9}) < 0$. So, $g_5(y, z) \geq h(z) \geq h(1) > 0.2$ ■

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